INTEGRAL TRANSFORMS OF FUNCTIONS TO BE IN THE PASCU CLASS USING DUALITY TECHNIQUES

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Abstract. Let $W_\beta(\alpha,\gamma)$, $\beta<1$, denote the class of all normalized analytic functions f in the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ such that

$$\operatorname{Re}\left(e^{i\phi}\left((1-\alpha+2\gamma)\frac{f}{z}+(\alpha-2\gamma)f'+\gamma zf''-\beta\right)\right)>0,\quad z\in\mathbb{D},$$

for some $\phi\in\mathbb{R}$ with $\alpha\geq 0$, $\gamma\geq 0$ and $\beta<1$. Let $M(\xi)$, $0\leq\xi\leq 1$, denote the Pascu class of ξ-convex functions given by the analytic condition

$$\operatorname{Re}\frac{\xi z(zf'(z))'+(1-\xi)zf'(z)}{\xi zf'(z)+(1-\xi)f(z)}>0$$

which unifies the class of starlike and convex functions. The aim of this paper is to find conditions on $\lambda(t)$ so that the integral transforms of the form

$$V_\lambda(f)(z)=\int_0^1\lambda(t)\frac{f(tz)}{t}dt.$$

carry functions from $W_\beta(\alpha,\gamma)$ into $M(\xi)$. As applications, for specific values of $\lambda(t)$, it is found that several known integral operators carry functions from $W_\beta(\alpha,\gamma)$ into $M(\xi)$. Results for a more generalized operator related to $V_\lambda(f)(z)$ are also given.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions f analytic in the open unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ with the normalization $f(0)=f'(0)-1=0$ and $\mathcal{S}$ be the class of functions $f\in\mathcal{A}$ that are univalent in $\mathbb{D}$. A function $f\in\mathcal{S}$ is said to be starlike (S^*) or convex (C), if f maps $\mathbb{D}$ conformally onto the domains, respectively, starlike with respect to origin and convex. Note that in $\mathbb{D}$, if $f\in C\iff zf'\in S^*$ follows from the well-known Alexander theorem (see [5] for details). An useful generalization of the class S^* is the class $S^*(\sigma)$ that has the analytic characterisation $S^*(\sigma)=\left\{f\in A:\operatorname{Re}\dfrac{zf'}{f}>\sigma;\ 0\leq\sigma<1\right\}$ and $S^*(0)\equiv S^*$. Various generalization of classes S^* and C are abundant in the literature. One such generalization is the following:

A function $f\in\mathcal{A}$ is said to be in the Pascu class of α -convex functions of order σ if [8]

$$\operatorname{Re}\frac{\alpha z(zf'(z))'+(1-\alpha)zf'(z)}{\alpha zf'(z)+(1-\alpha)f(z)}>\sigma,\qquad 0\leq\alpha\leq 1,\quad 0\leq\alpha\leq 1,$$

or in other words

$$\alpha zf'(z)+(1-\alpha)f(z)\in\mathcal{S}^*(\sigma).$$

This class is denoted by $M(\alpha,\sigma)$. Even though, this class is known as Pascu class of α -convex functions of order σ, since we use the parameter α for another important class, we denote this class by $M(\xi,\sigma)$, $0\leq\xi\leq 1$, and we remark that, in the sequel, we only

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consider the class $M(\xi) := M(\xi, 0)$. Clearly $M(0) = S^*$ and $M(1) = C$ which implies that this class $M(\xi)$ is a smooth passage between the class of starlike and convex functions.

The main objective of this work is to find conditions on the non-negative real valued integrable function $\lambda(t)$ satisfying $\int_0^1 \lambda(t)dt = 1$, such that the operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}dt \tag{1.1}$$

is in the class $M(\xi)$. Note that this operator was introduced in [6]. To investigate this admissibility property the class to which the function f belongs is important. Let $W_\beta(\alpha, \gamma)$, $\alpha \geq 0$, $\gamma \geq 0$ and $\beta < 1$, denote the class of all normalized analytic functions f in the unit disc $\mathbb{D}$ such that

$$\mathrm{Re}\left(e^{i\phi}\left((1-\alpha+2\gamma)\frac{f}{z} + (\alpha - 2\gamma)f' + \gamma z f'' - \beta\right)\right) > 0, \quad z \in \mathbb{D}$$

for some $\phi \in \mathbb{R}$. This class and its particular cases were considered by many authors so that the corresponding operator given by (1.1) is univalent and in $M(\xi)$ for some particular values of α, β, γ and ξ. This work was motivated in [6] by studying the conditions under which $V_\lambda(W_\beta(1, 0)) \subset M(0)$ and generalized in [7] by studying the case $V_\lambda(W_\beta(\alpha, 0)) \subset M(0)$. Similar situation for the convex case, namely $V_\lambda(W_\beta(1, 0)) \subset M(1)$ was initiated in [4]. After several generalizations by many authors, recently, the conditions under which $V_\lambda(W_\beta(\alpha, \gamma)) \subset M(0)$ was obtained in [1] and the corresponding results for the convex case so that $V_\lambda(W_\beta(\alpha, \gamma)) \subset M(1)$ was obtained in [3]. Applications involving several well known integral transforms were studied in [1] and [3] (see also [11]) For all the literature involving the complete study in this direction so far we refer to [1, 3, 9, 11] and references therein.

In this work, we find conditions on $\lambda(t)$ so that $V_\lambda(W_\beta(\alpha, \gamma)) \subset M(\xi)$ using duality techniques which are presented in Section 2. As applications, in Section 3, we consider particular values for $\lambda(t)$ in (1.1) so that results for some of the well-known integral operators can be deduced. A more generalized operator introduced in [4] is considered in Section 4 for similar type of results.

First we underline some preliminaries that are useful for our discussion. We introduce two constants $\mu \geq 0$ and $\nu \geq 0$ satisfying [1, 2]

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma. \tag{1.2}$$

When $\gamma = 0$, then μ is chosen to be 0, in which case, $\nu = \alpha \geq 0$. When $\alpha = 1 + 2\gamma$, (1.2) yields $\mu + \nu = 1 + \gamma = 1 + \mu\nu$, or $(\mu - 1)(1 - \nu) = 0$, and leads to two cases

(i) For $\gamma > 0$, then choosing $\mu = 1$ gives $\nu = \gamma$.
(ii) For $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha = 1$.

Note 1. *Since the case $\gamma = 0$ is considered in* [9], *we only consider results for the case $\gamma > 0$, except for Theorem 3.2 (see Remark 3.2).*

Next we introduce two known auxiliary functions [1]. Let

$$\phi_{\mu,\nu}(z) = 1 + \sum_{n=1}^{\infty} \frac{(n\nu + 1)(n\mu + 1)}{n + 1} z^n, \tag{1.3}$$

and

$$\psi_{\mu,\nu}(z) = \phi_{\mu,\nu}^{-1}(z) \;=\; 1 + \sum_{n=1}^{\infty} \frac{n + 1}{(n\nu + 1)(n\mu + 1)} z^n = \int_0^1 \int_0^1 \frac{ds dt}{(1 - t^\nu s^\mu z)^2}. \tag{1.4}$$

Here $\phi_{\mu,\nu}^{-1}$ denotes the convolution inverse of $\phi_{\mu,\nu}$ such that $\phi_{\mu,\nu} * \phi_{\mu,\nu}^{-1} = 1/(1-z)$. By $*$, we mean the following: If f and g are in $\mathcal{A}$ with the power series expansions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ respectively, then the convolution or Hadamard product of f and g is given by $h(z) = \sum_{k=0}^{\infty} a_k b_k z^k$.

Since $\nu \geq 0$, $\mu \geq 0$, when $\gamma \geq 0$, making the change of variables $u = t^\nu$, $v = s^\mu$ in (1.4) result in writing $\psi_{\mu,\nu}$ as

$$\psi_{\mu,\nu}(z) = \begin{cases} \dfrac{1}{\mu\nu}\displaystyle\int_0^1\int_0^1 \frac{u^{1/\nu-1}v^{1/\mu-1}}{(1-uvz)^2}dudv, & \gamma > 0, \\[2ex] \displaystyle\int_0^1 \frac{dt}{(1-t^\alpha z)^2}, & \gamma = 0, \alpha \geq 0. \end{cases}$$

Now let g be the solution of the initial value-problem

$$\frac{d}{dt}t^{1/\nu}(1+g(t)) = \begin{cases} \dfrac{2}{\mu\nu}t^{1/\nu-1}\displaystyle\int_0^1 \frac{s^{1/\mu-1}}{(1+st)^2}ds, & \gamma > 0, \\[2ex] \dfrac{2}{\alpha}\dfrac{t^{1/\alpha-1}}{(1+t)^2}, & \gamma = 0, \alpha > 0, \end{cases} \tag{1.5}$$

satisfying $g(0) = 1$. The series solution is given by

$$g(t) = 2\sum_{n=0}^{\infty} \frac{(n+1)(-1)^n t^n}{(1+\mu n)(1+\nu n)} - 1. \tag{1.6}$$

Let q be the solution of the differential equation

$$\frac{d}{dt}t^{1/\nu}q(t) = \begin{cases} \dfrac{1}{\mu\nu}t^{1/\nu-1}\displaystyle\int_0^1 s^{1/\mu-1}\frac{(1-st)}{(1+st)^3}ds, & \gamma > 0 \\[2ex] \dfrac{1}{\alpha}t^{1/\alpha-1}\dfrac{(1-t)}{(1+t)^3}, & \gamma = 0, \alpha > 0. \end{cases} \tag{1.7}$$

satisfying $q(0) = 0$. The series solution of $q(t)$ is given by

$$q(t) = \sum_{n=0}^{\infty} \frac{(n+1)^2(-1)^n t^n}{(1+\mu n)(1+\nu n)} \tag{1.8}$$

Note that $q(t)$ also satisfies $2q(t) = tg'(t) + g(t) + 1$.

Our main results is the generalization of the following results given in [1] and [3]. The necessary and sufficient conditions under which the operator $V_\lambda(f(z))$ carries the function $f(z)$ from $W_\beta(\alpha,\gamma)$, to the classes S^* and C, respectively are given in next two results.

Theorem 1.1. [1] *Consider $\mu \geq 0$, $\nu \geq 0$ given by* (1.2) *with $\beta < 1$ satisfying*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)g(t)dt, \tag{1.9}$$

where g is the solution of the initial value-problem (1.5) *and let $f \in \mathcal{W}_\beta(\alpha,\gamma)$. Assume that $t^{1/\nu}\Lambda_\nu(t) \to 0$, and $t^{1/\mu}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Then $F(z) = V_\lambda(f)(z)$ is in $\mathcal{S}^*$ if*

and only if

$$\begin{cases} \quad \mathrm{Re}\displaystyle\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(\frac{h(tz)}{tz}-\frac{1}{(1+t)^2}\right)dt \geq 0, \quad \gamma>0, \\ \quad \mathrm{Re}\displaystyle\int_0^1 \Pi_{0,\alpha}(t)t^{1/\alpha-1}\left(\frac{h(tz)}{tz}-\frac{1}{(1+t)^2}\right)dt \geq 0, \quad \gamma=0, \end{cases} \tag{1.10}$$

where

$$\Lambda_\nu(t)=\int_t^1 \frac{\lambda(x)}{x^{1/\nu}}dx, \quad \nu>0, \tag{1.11}$$

$$\Pi_{\mu,\,\nu}(t)=\begin{cases} \quad \displaystyle\int_t^1 \Lambda_\nu(x)x^{1/\nu-1-1/\mu}dx, \quad \gamma>0(\mu>0,\nu>0), \\ \quad \Lambda_\alpha(t), \quad \gamma=0(\mu=0,\nu=\alpha>0) \end{cases} \tag{1.12}$$

and

$$h(z)=\frac{z(1+\frac{\epsilon-1}{2}z)}{(1-z)^2}, \quad |\epsilon|=1.$$

Theorem 1.2. [3] *Let $f\in\mathcal{W}_\beta(\alpha,\gamma)$, $\mu\geq 0$, $\nu\geq 0$ satisfy* (1.2) *and $\beta<1$, be given by*

$$\frac{\beta-1/2}{1-\beta}=-\int_0^1 \lambda(t)q(t)dt, \tag{1.13}$$

where q is given by (1.7)*. Further $\Lambda_\nu(t)$ and $\Pi_{\nu,\,\mu}(t)$ are given in* (1.11) *and* (1.12) *and assume that $t^{1/\mu}\Lambda_\nu(t)\to 0$, and $t^{1/\nu}\Pi_{\mu,\,\nu}(t)\to 0$ as $t\to 0^+$. Then*

$$\begin{cases} \quad \mathrm{Re}\displaystyle\int_0^1 \Pi_{\nu,\,\mu}(t)t^{1/\mu-1}\left(h'(tz)-\frac{1-t}{(1+t)^3}\right)dt \geq 0, \quad \gamma>0, \\ \quad \mathrm{Re}\displaystyle\int_0^1 \Pi_{0,\,\alpha}(t)t^{1/\alpha-1}\left(h'(tz)-\frac{1-t}{(1+t)^3}\right)dt \geq 0, \quad \gamma=0, \end{cases} \tag{1.14}$$

if and only if $F(z)=V_\lambda(f)(z)$ is in C.

It is difficult to verify conditions (1.10) and (1.14). Hence the following results involving sufficient conditions are useful for finding applications.

Theorem 1.3. [1] *Let Λ_ν and $\Pi_{\mu,\nu}$ are defined in* (1.11) *and* (1.12)*. Assume that both $\Pi_{\mu,\nu}$ and Λ_ν are integrable on $[0,1]$ and positive on $(0,1)$. Assume further that $\mu\geq 1$ and*

$$\frac{\Pi_{\mu,\nu}(t)}{1-t^2} \tag{1.15}$$

is decreasing on $(0,1)$. If β satisfies (1.9) *and $f\in\mathcal{W}_\beta(\alpha,\gamma)$, then $V_\lambda(f)\in\mathcal{S}^*$.*

Theorem 1.4. [3] *Let Λ_ν and $\Pi_{\mu,\nu}$ are defined in* (1.11) *and* (1.12)*. Assume that both are integrable on $[0,1]$ and positive on $(0,1)$. Assume further that $\mu\geq 1$ and*

$$\frac{\Lambda_\nu(t)t^{1/\nu-1/\mu}+(1-1/\mu)\Pi_{\mu,\nu}(t)}{1-t^2}, \tag{1.16}$$

is decreasing on $(0,1)$. If β satisfies (1.13) *and $f\in\mathcal{W}_\beta(\alpha,\gamma)$, then $V_\lambda(f)\in C$.*

2. Main Results

We start with a result that gives both necessary and sufficient condition for an integral transform that satisfies the admissibility property of the class $W_\beta(\alpha,\gamma)$, which contain non-univalent functions also, to the Pascu class $M(\xi)$.

Theorem 2.1. *Let $\mu > 0$, $\nu > 0$, satisfies* (1.2) *and $\beta < 1$ satisfies*

$$\frac{\beta}{(1-\beta)} = -\int_0^1 \lambda(t)[(1-\xi)g(t)+\xi(2q(t)-1)]dt, \tag{2.1}$$

where $g(t)$ and $q(t)$ are defined by the differential equations given in (1.5) *and* (1.7) *respectively. Assume that $t^{1/\nu}\Lambda_\nu(t) \to 0$ and $t^{1/\mu}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Then*

$$N_{\Pi_{\mu,\nu}} \geq 0 \Longleftrightarrow F = V_\lambda(W_\beta(\alpha,\gamma)) \in M(\xi)$$

or

$$\xi z F' + (1-\xi)F \in S^*,$$

where

$$N_{\Pi_{\mu,\nu}}(h) = \inf_{z\in\Delta} \int_0^1 t^{1/\mu-1}\Pi_{\mu,\nu}(t)\mathcal{L}_{\xi,z}(t)dt \tag{2.2}$$

and

$$\mathcal{L}_{\xi,z}(t) = (1-\xi)\left(\mathrm{Re}\,\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right) + \xi\left(\mathrm{Re}\; h'(tz) - \frac{(1-t)}{(1+t)^3}\right).$$

The value of β is sharp.

Proof. Let

$$H(z) = (1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) \quad \text{and} \quad G(z) = \frac{(H(z)-\beta)}{1-\beta}.$$

Since Re $e^{i\phi}G(z) > 0$, [10] we may assume that $G(z) = \dfrac{1+xz}{1+yz}$, $|\, x \,| = \,|\, y \,| = 1$. Further, assuming $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, from (1.2) we get

$$\begin{aligned} H(z) &= (1+\mu\nu-\nu-\mu)\,\frac{f(z)}{z} + (\nu+\mu-\mu\nu)\,f'(z) + \mu\nu z f''(z) \\ &= \mu\nu z^{1-1/\mu}\frac{d}{dz}\left[z^{1/\mu-1/\nu+1}\frac{d}{dz}\left(z^{1/\nu-1}f(z)\right)\right] = f'(z) * \phi_{\mu,\nu}(z) \\ \Longrightarrow \qquad f'(z) &= H(z) * \psi_{\mu,\nu} = \left[(1-\beta)\left(\frac{1+xz}{1+yz}\right)+\beta\right] * \psi_{\mu,\nu}, \end{aligned}$$

using (1.3) and (1.4). This gives

$$\frac{f(z)}{z} = \frac{1}{z}\int_0^z \left[(1-\beta)\left(\frac{1+x\omega}{1+y\omega}\right)+\beta\right] d\omega * \psi_{\mu,\nu}. \tag{2.3}$$

Since $F \in M(\xi)$, implies $\xi zF' + (1-\xi)F \in S^*$, by the well known result [10, p.94] of convolution theory we get

$$\xi zF' + (1-\xi)F \in S^* \quad \text{if and only if} \quad 0 \neq \frac{1}{z}\left[((1-\xi)F + \xi zF') * h(z)\right], \qquad |z| < 1.$$

Thus

$$
\begin{aligned}
0 &\neq \left((1-\xi)\frac{F}{z}+\xi F'\right)*\frac{h(z)}{z}\\
&= (1-\xi)\int_0^1 \frac{\lambda(t)}{1-tz}*\frac{f(z)}{z}*\frac{h(z)}{z}+\xi\int_0^1 \frac{\lambda(t)}{1-tz}*f'(z)*\frac{h(z)}{z}\\
&= (1-\xi)\int_0^1 \lambda(t)\frac{h(tz)}{tz}dt*\left[\frac{1}{z}\int_0^z(1-\beta)\left(\frac{1+x\omega}{1+y\omega}\right)d\omega+\beta\right]\\
&\quad +\xi\int_0^1 \lambda(t)h'(tz)dt*\left[\frac{1}{z}\int_0^z(1-\beta)\left(\frac{1+x\omega}{1+y\omega}\right)d\omega+\beta\right]\\
&= (1-\beta)\left[(1-\xi)\int_0^1\lambda(t)\left(\frac{1}{z}\int_0^z\frac{h(t\omega)}{t\omega}d\omega+\frac{\beta}{1-\beta}\right)dt\right.\\
&\quad \left.+\xi\int_0^1\lambda(t)\left(\frac{1}{z}\int_0^z h'(t\omega)d\omega+\frac{\beta}{1-\beta}\right)dt\right]*\psi(z)*\frac{1+xz}{1+yz}.
\end{aligned}
$$

This condition holds if and only if [10, p.23]

$$
\begin{aligned}
\operatorname{Re}&\left((1-\beta)\left[(1-\xi)\int_0^1\lambda(t)\left(\frac{1}{z}\int_0^z\frac{h(t\omega)}{t\omega}d\omega+\frac{\beta}{1-\beta}\right)dt\right.\right.\\
&\left.\left.+\xi\int_0^1\lambda(t)\left(\frac{1}{z}\int_0^z h'(t\omega)d\omega+\frac{\beta}{1-\beta}\right)dt\right]*\psi(z)\right)>\frac{1}{2}.
\end{aligned}
$$

Using the results (1.10) and(1.14) from Theorem 1.1 and Theorem 1.2 respectively, we obtain

$$
\int_0^1 t^{1/\mu-1}\Pi_{\mu,\nu}(t)\left[(1-\xi)\left(\operatorname{Re}\frac{h(tz)}{tz}-\frac{1}{(1+t)^2}\right)+\xi\left(\operatorname{Re}h'(tz)-\frac{(1-t)}{(1+t)^3}\right)\right]dt\geq 0
$$

which is the required result.

To verify sharpness, let $W_\beta(\alpha,\gamma)$ be the solution of the differential equation

$$
(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)=\beta+(1-\beta)\frac{1+z}{1-z}
$$

where $\beta<\beta_0$ satisfies (2.1). Further simplification using (1.6) and (1.8) gives

$$
\frac{\beta_0}{1-\beta_0}=-1-\left[(1-\xi)\sum_{n=1}^{\infty}\frac{2(n+1)(-1)^n\tau_n}{(1+\mu n)(1+\nu n)}+\xi\sum_{n=1}^{\infty}\frac{2(n+1)^2(-1)^n\tau_n}{(1+\mu n)(1+\nu n)}\right],
$$

where $\tau_n=\displaystyle\int_0^1\lambda(t)t^n dt$.

Clearly $F=V_\lambda(f(z))\in M(\xi)\implies K(z):=\xi zF'+(1-\xi)F\in S^*$. Using $f'(z)=H(z)*\psi_{\mu,\nu}$ and the series expansion of $f(z)$, we get

$$
f(z)=z+\sum_{n=1}^{\infty}\frac{2(1-\beta)}{(n\mu+1)(n\nu+1)}z^{n+1}.
$$

This means

$$F = V_\lambda(f(z)) = z + \sum_{n=1}^{\infty} \frac{2(1-\beta)}{(n\mu+1)(n\nu+1)} \left(\int_0^1 \lambda(t) t^n dt \right) z^{n+1}$$

$$= z + \sum_{n=1}^{\infty} \frac{2(1-\beta)\tau_n}{(n\mu+1)(n\nu+1)} z^{n+1}. \tag{2.4}$$

Using (2.4), a simple computation gives

$$K(z) = (1-\xi)\left(z + \sum_{n=1}^{\infty} \frac{2(1-\beta)\tau_n z^{n+1}}{(n\mu+1)(n\nu+1)} \right) + \xi \left(z + \sum_{n=1}^{\infty} \frac{2(1-\beta)(n+1)\tau_n z^{n+1}}{(n\mu+1)(n\nu+1)} \right).$$

This means

$$zK'(z)|_{z=-1} = 1 + \xi \sum_{n=1}^{\infty} \frac{2(1-\beta)(n+1)^2\tau_n(-1)^n}{(n\mu+1)(n\nu+1)} + (1-\xi)\sum_{n=1}^{\infty} \frac{2(1-\beta)(n+1)\tau_n(-1)^n}{(n\mu+1)(n\nu+1)}$$

$$= 1 - \frac{(1-\beta)}{1-\beta_0} < 0.$$

Hence $zK'(z) = 0$ for some $z \in \mathbb{D}$, so $K(z)$ is not even locally univalent in $\mathbb{D}$. This shows that the result is sharp for β. □

Remark 2.1. *This result generalizes various results known in this direction. For example, $\xi = 0$ gives Theorem* 1.1 [1, Theorem 3.1] *and $\xi = 1$ gives Theorem* 1.2 [3, Theorem 3.1]. *For other particular cases with $\xi = 0$ or* 1, *we refer to* [1, 3] *and references therein.*

Theorem 2.2. *Let $\Pi_{\mu,\nu}$ and Λ_ν be defined as in* (1.11) *and* (1.12) *respectively, with both of them integrable on* $[0,1]$ *and positive on* $(0,1)$. *Further assume that $0 \le \xi \le 1$, $\mu \ge 1$ and*

$$\frac{\xi t^{1/\xi - 1/\mu + 1} \; d\big(t^{1/\mu - 1/\xi}\Pi_{\mu,\nu}(t)\big)}{(1-t^2)} \tag{2.5}$$

is increasing on $(0,1)$. *Then for β satisfying* (2.1), $V_\lambda(W_\beta(\alpha,\gamma)) \in M(\xi)$.

Proof. Consider

$$N_{\Pi_{\mu,\nu}}(h) = \int_0^1 t^{1/\mu-1}\Pi_{\mu,\nu}(t)\left[(1-\xi)\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2} \right) + \xi \left(h'(tz) - \frac{(1-t)}{(1+t)^3} \right) \right] dt$$

$$= \int_0^1 t^{1/\mu-1}\Pi_{\mu,\nu}(t)\left[(1-\xi)\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2} \right) + \xi \frac{d}{dt}\left(\frac{h(tz)}{z} - \frac{t}{(1+t)^2} \right) \right] dt.$$

Integration by parts gives

$$N_{\Pi_{\mu,\nu}}(h) = (1-\xi)\int_0^1 t^{1/\mu-1}\Pi_{\mu,\nu}(t)\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2} \right) dt$$

$$-\xi \int_0^1 \frac{d}{dt}\left(t^{1/\mu-1}\Pi_{\mu,\nu}(t) \right)\left(\frac{h(tz)}{z} - \frac{t}{(1+t)^2} \right) dt$$

$$= \int_0^1 \left(\left[(1-\xi) - \xi(\frac{1}{\mu} - 1) \right] \Pi_{\mu,\nu}(t) t^{1/\mu-1} - \xi \Pi'_{\mu,\nu}(t) t^{1/\mu} \right)\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2} \right) dt.$$

$$= \int_0^1 t^{1/\mu-1}\left((1-\frac{\xi}{\mu})\Pi_{\mu,\nu}(t) - \xi t \Pi'_{\mu,\nu}(t) \right)\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2} \right) dt,$$

by a simple computation. It is easy to see that, from Theorem (1.3) and (1.4), $N_{\Pi_{\mu,\nu}}(h) \geq 0$ only when $\xi \dfrac{\Pi_{\mu,\nu}(t)}{1-t^2} + (1-\xi)\dfrac{\Lambda_\nu(t)t^{1/\nu-1/\mu} + (1-1/\mu)\Pi_{\mu,\nu}(t)}{1-t^2}$ is decreasing on $(0,1)$ which is nothing but (2.5) and the proof is complete by applying Theorem 2.1. □

Remark 2.2. *Even though, we did not consider the case $\gamma = 0$, even at $\gamma = 0$, Theorem 2.2 does not reduces to a similar result given in* [9]. *This is due to the fact that, our condition* (2.5) *has the term $(1-t^2)$ in the denominator, whereas the corresponding result in* [9] *has the term $\log(1/t)$ and hence has different condition.*

3. Applications

It is difficult to check the condition given in Section 2, for $V_\lambda(W_\beta(\alpha,\gamma)) \subset M(\xi)$. In order to find applications, simplified conditions are required. For this purpose, from (2.5), it is enough to show that

$$\frac{\xi t^{1/\xi-1/\mu+1} \;\; d\big(t^{1/\mu-1/\xi}\Pi_{\mu,\nu}(t)\big)}{(1-t^2)}$$

is increasing on (0,1) which is equivalent of having

$$g(t) = \frac{\left(1-\dfrac{\xi}{\mu}\right)\Pi_{\mu,\nu}(t) + \xi t^{1/\nu-1/\mu}\Lambda_\nu(t)}{(1-t^2)}$$

is decreasing on (0,1), where $\Lambda_\nu(t)$ and $\Pi_{\mu,\nu}(t)$ are defined in (1.11) and (1.12). It is enough to have $g'(t) \leq 0$.

Let $g(t) = p(t)/(1-t^2)$ where $p(t) = \left(1-\dfrac{\xi}{\mu}\right)\Pi_{\mu,\nu}(t) + \xi t^{1/\nu-1/\mu}\Lambda_\nu(t)$. So to satisfy the above condition we need to have

$$L(t) = p(t) + \frac{(1-t^2)p'(t)}{2t} \leq 0.$$

Since $\Lambda_\nu(1) = 0$ and $\Pi_{\mu,\nu}(1) = 0$ we get $L(1) = 0$. This implies that it suffies to have $L(t)$ is increasing on (0,1), which means

$$L'(t) = \frac{(1-t^2)}{2t^2}[tp''(z) - p'(z)] \geq 0.$$

The above equation also holds if $tp''(z) - p'(z) \geq 0$ which is equivalent to the condition

$$\left(\frac{\xi}{\nu}-1\right)\left(\frac{1}{\nu}-\frac{1}{\mu}-2\right)\Lambda_\nu(t) + \left(1+\xi\left[1+\frac{1}{\mu}-\frac{1}{\nu}\right]\right)t^{1-1/\nu}\lambda(t) - \xi t^{2-1/\nu}\lambda'(t) \geq 0. \tag{3.1}$$

This inequality can further be reduced to

$$(1-\xi)\left[\left(1+\frac{1}{\mu}\right)\lambda(t) - t\lambda'(t)\right] + \xi\left[t^2\lambda''(t) - \frac{1}{\mu}t\lambda'(t)\right] \geq 0 \tag{3.2}$$

if the inequality

$$\xi\frac{\lambda'(1)}{\lambda(1)} \leq 1 + \xi\left(1+\frac{1}{\mu}-\frac{1}{\nu}\right) \tag{3.3}$$

is true. So, in order to obtain further results, we check conditions (3.2) and (3.3). Note that, whenever $\lambda(1) = \lambda'(1) = 0$, from (3.1) we see that it is sufficient to check (3.2) as there is no necessity for the condition given by (3.3).

As the first application, we consider $\lambda(t)=(c+1)t^c, c>-1$, so that the Bernardi operator of function in $W_\beta(\alpha,\gamma)$ is in $M(\xi)$.

Theorem 3.1. *Let $0\leq\xi\leq 1$, $\nu\geq\mu\geq 1$ and $\beta<1$ satisfies (2.1). If $f(z)\in W_\beta(\alpha,\gamma)$, then the function, given by the Bernardi operator,*

$$V_\lambda(f)(z)=(1+c)\int_0^1 t^{c-1}f(tz)dt$$

belongs to $M(\xi)$ if

$$-1<c\leq min\left[\left(1+\frac{1}{\mu}-\frac{1}{\nu}\right),\left(\frac{1+\frac{1}{\mu}-\xi}{1+2\xi}\right)\right]. \tag{3.4}$$

Proof. With $\lambda(t)=(1+c)t^c$, using (3.4), we get

$$\xi\frac{\lambda'(1)}{\lambda(1)}=\xi c\quad\leq\xi\left(1+\frac{1}{\mu}-\frac{1}{\nu}\right)\quad\leq 1+\xi\left(1+\frac{1}{\mu}-\frac{1}{\nu}\right).$$

So the inequality (3.3) is satisfied. If the inequality (3.2) holds then $V_\lambda(f)(z)\in M(\xi)$ which means

$$(1-\xi)\left[\left(1+\frac{1}{\mu}\right)(c+1)-c(c+1)\right]+\xi\left[(c-1)c(c+1)-\frac{1}{\mu}c(c+1)\right]\geq 0.$$

On further simplification this inequality reduces to

$$1+\frac{1}{\mu}-c+c^2\xi\geq 0. \tag{3.5}$$

Clearly $(c+1)^2>0$. Hence using $c^2>-(2c+1)$ and substituting in (3.5), we get

$$\left(c^2\xi+1+\frac{1}{\mu}-c\right)>\left(-c(2\xi+1)+1+\frac{1}{\mu}-\xi\right),$$

which is true by the hypothesis. □

Remark 3.1.

(1) *When $\xi=0$ then $-1<c\leq min\left[\left(1+\frac{1}{\mu}-\frac{1}{\nu}\right),\left(1+\frac{1}{\mu}\right)\right]=\left(1+\frac{1}{\mu}-\frac{1}{\nu}\right)$ whereas in* [1]*, the range for c is given as $-1<c\leq\left(1+\frac{1}{\mu}\right)$.*

(2) *For $\xi=1$, we have $-1<c\leq min\left[\left(1+\frac{1}{\mu}-\frac{1}{\nu}\right),\left(\frac{1}{3\mu}\right)\right]=\left(\frac{1}{3\mu}\right)$. Result obtained in* [3]*, for $\nu\geq\mu\geq 1$ is $-1<c\leq\left(2+\frac{1}{\mu}-\frac{1}{\nu}\right)$.*

So in both the cases the result (Theorem 3.1) obtained for admissibility property of Bernardi operator in $M(\xi)$ class differs from [1, Theorem 5.1] and [3, Theorem 5.1]. But our result is true for $0<\xi<1$ also.

Remark 3.2. *We recall that, throughout this paper we use $\gamma>0$, as $\gamma=0$ case is considered in* [9]*. But we note that, for the Bernardi operator, the result given in* [9] *uses the fact $\lambda(1)=0$, which is not true. Hence , in order to make completion of the work for $W_\beta(\alpha,\gamma)$ in this direction we give the result related to Bernardi operator for $\gamma=0$. Since the condition for $\lambda(t)$, given in* [9] *is different from the one given by* (3.3) *and* (3.2)*, we explicitly prove this result.*

Theorem 3.2. *Let $0 < \xi < 1$, $\mu \geq 1$ and $\beta < 1$ satisfies* (2.1)*. If $f(z) \in W_\beta(\alpha, \gamma)$, then the function*

$$V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1} f(tz)dt \tag{3.6}$$

belongs to $M(\xi)$ if $c > 1 + \dfrac{1}{\alpha}$ and $\xi \geq \alpha$ for $\gamma = 0$.

Proof. Note that for the Bernardi operator, the case $\xi = 0$ is considered in [1, Theorem. 5.1] and $\xi = 1$ is considered in [3, Theorem. 5.1]. We consider only the case $0 < \xi < 1$. For the case $\gamma = 0$, the integral operator $V_\lambda(W_\beta(\alpha, \gamma))$ is in $M(\xi)$ if and only if

$$\frac{\left(1 - \dfrac{\xi}{\alpha}\right)\Pi_{0,\alpha}(t) + \xi t^{1-1/\alpha}\lambda(t)}{(1-t^2)} \quad \text{is decreasing on} \quad (0,1),$$

which can be easily obtained as in Theorem 2.2.
Let $J(t) = \dfrac{p(t)}{(1-t^2)}$, where

$$p(t) = \left(1 - \frac{\xi}{\alpha}\right)\Pi_{0,\alpha}(t) + \xi t^{1-1/\alpha}\lambda(t)$$

for the integral operator to be in $M(\xi)$, $J'(t) \leq 0 \implies p(t) + \dfrac{(1-t^2)}{2t}p'(t) \leq 0$
Substituting the value of $p(t)$ and $p'(t)$ in the above equation, we have
$J(t)$

$$= \left(1 - \frac{\xi}{\alpha}\right)\Pi_{0,\alpha}(t) + \frac{(\xi+1)}{2}t^{1-1/\alpha}\lambda(t) + \frac{(\xi-1)}{2}t^{-1-1/\alpha}\lambda(t) + \frac{\xi}{2}t^{-1/\alpha}(1-t^2)\lambda'(t) \leq 0 \tag{3.7}$$

Consider the case when $\lambda(t) = (c+1)t^c$, where $c > -1$. So the inequality (3.7) on simplification reduces to

$$\begin{aligned} J(t) = \left(1 - \frac{\xi}{\alpha}\right)\frac{c+1}{c - \dfrac{1}{\alpha} + 1}\left[1 - t^{c-1/\alpha+1}\right] + \frac{1}{2}(c+1)(\xi + 1 - c\xi)t^{c+1-1/\alpha} \\ + \frac{1}{2}(c+1)(\xi - 1 + c\xi)t^{c-1-1/\alpha} \leq 0. \end{aligned} \tag{3.8}$$

If $c > 1 + \dfrac{1}{\alpha}$ and $\alpha \leq \xi$, then for $t = 0$, (3.8) is satisfied.
For $c > 1 + \dfrac{1}{\alpha}$, $J(0) \leq 0$ which clearly implies that if $J(t)$ is decreasing on $t \in (0,1)$ i.e., $J'(t) \leq 0$, then also the inequality (3.8) holds. Further simplifying (3.7), for Bernardi operator, we need to show that

$$\frac{1}{2}(1-\xi)\left(1 + \frac{1}{\alpha}\right)\lambda(t) + \frac{1}{2}\left[\xi\left(1 - \frac{1}{\alpha}\right) - 1\right]t\lambda'(t) + \frac{\xi}{2}t^2\lambda''(t) \leq 0. \tag{3.9}$$

On substituting the values of $\lambda(t)$, $\lambda'(t)$ and $\lambda''(t)$ in (3.9), we get

$$\left(1 - c - \frac{1}{\alpha}\right) - \xi\left(1 - c^2 - \frac{1}{\alpha} + \frac{c}{\alpha}\right) \leq 0. \tag{3.10}$$

To satisfy (3.10), ξ should hold

$$\xi \geq \frac{\alpha(1-c)-1}{(c-1)+\alpha(1-c^2)}.$$

Since $\alpha \leq \xi$, we get

$$\xi \geq \max\left(\alpha,\, \frac{\alpha(1-c)-1}{(c-1)+\alpha(1-c^2)}\right) = \alpha$$

satisfying the hypothesis of theorem which gives $V_\lambda(f)(z)$ given by (3.6) is in $M(\xi)$. $\square$

Theorem 3.3. *Let $0 \leq \xi \leq 1$ and $\nu \geq \mu \geq 1$. If $F \in \mathcal{A}$ satisfies,*

$$\mathrm{Re}\,(F'(z) + \alpha z F''(z) + \gamma z^2 F'''(z)) > \beta$$

in $\mathbb{D}$, and $\beta < 1$ satisfies

$$\frac{\beta}{(1-\beta)} = -\int_0^1 \lambda(t)[(1-\xi)g(t) + \xi(2q(t)-1)]dt,$$

where $g(t)$ and $q(t)$ are given by (1.6) *and* (1.8) *respectively. Then $F \in M(\xi)$.*

Proof. Let $f(z) = zF'(z)$, then $f \in W_\beta(\alpha, \gamma)$. Therefore

$$F(z) = \int_0^1 \frac{f(tz)}{t} dt.$$

When $c = 0$, the hypothesis $\nu \geq \mu \geq 1$ satisfies (3.4) and hence from Theorem 3.1, the required result follows. $\square$

Example 3.1. *If $\gamma = 1$, $\alpha = 3$, then $\mu = 1 = \nu$. In this case,* (1.6) *and* (1.8) *yield*

$$\begin{aligned}\frac{\beta}{(1-\beta)} &= 2(1-\xi)\int_0^{-1} \frac{\log(1-t)}{t} dt - 2\xi \log 2 + 1 \\ &= 1 - 2(1-\xi)\left(\frac{\pi^2}{12}\right) - 2\xi \log 2.\end{aligned}$$

Thus Re $(f'(z) + 3zf''(z) + z^2 f'''(z)) > \beta \Longrightarrow f \in M(\xi)$.

Remark 3.3.

(1) *for $\xi = 0$* [1, Remark 5.2], *we get $\beta = -1.816378$, such that $f \in M(0)$.*
(2) *For $\xi = 1$* [3, Example 5.2], *we get $\beta = -0.629445$, such that $f \in M(1)$.*

Theorem 3.4. *Let $0 \leq \xi \leq 1$, $\mu \geq 1$, $B < 1$ and $\beta < 1$ satisfies* (1.13). *If $f(z) \in W_\beta(\alpha, \gamma)$, then the function*

$$V_\lambda(f)(z) = k\int_0^1 t^{B-1}(1-t)^{C-A-B}\phi(1-t)\frac{f(tz)}{t}dt$$

belongs to $M(\xi)$ if

$$B < min\left(\left[2 + \frac{1}{\mu}\right], (C-A-1)\right).$$

Proof. For $V_\lambda(f)(z)$ to be in $M(\xi)$, $\lambda(t)$ should satisfy (3.1), where
$\lambda(t) = kt^{B-1}(1-t)^{C-A-B}\phi(1-t)$ and
$\lambda'(t)$

$$= Kt^{B-2}(1-t)^{C-A-B-1}\left[((B-1)(1-t) - (C-A-B)t)\,\phi(1-t) - t(1-t)\phi'(1-t)\right],$$

Since $C-A-B>1$, $\lambda(1)=0$ and $\lambda'(1)=0$, generates that it is enough to check (3.2). If we substitute the values of $\lambda(t)$, $\lambda'(t)$ and $\lambda''(t)$ in (3.2), where

and

$$\begin{aligned}\lambda''(t)=&Kt^{B-3}(1-t)^{C-A-B-2}[((B-1)(B-2)(1-t)^2\\&-2(B-1)(C-A-B)t(1-t)+(C-A-B)(C-A-B-1)t^2)\phi(1-t)\\&+[2(C-A-B)t-2(B-1)(1-t)]t(1-t)\phi'(1-t)+t^2(1-t)^2\phi"(1-t)].\end{aligned}$$

a simple computation shows that (3.2) is equivalent to

$$(1-\xi)h_1(t)+\xi h_2(t)\geq 0,$$

where

$$h_1(t)=kt^{B-1}(1-t)^{C-A-B-1}(X_1(t)\phi(1-t)-t(1-t)\phi'(1-t))$$

with

$$X_1(t)=\left[\left(1+\frac{1}{\mu}\right)(1-t)-(B-1)(1-t)+(C-A-B)t\right]$$

and

$$h_2(t)=kt^{B-1}(1-t)^{C-A-B-2}(X_2(t)\phi(1-t)+X_3(t)\phi'(1-t)+X_4(t)\phi''(1-t))$$

with

$$\begin{aligned}X_2(t)=&(B-1)(1-t)^2\left[B-2-\frac{1}{\mu}\right]+(C-A-B)t(1-t)\left[\frac{1}{\mu}-2(B-1)\right]\\&+(C-A-B)(C-A-B-1)t^2,\end{aligned}$$

$$X_3(t)=[2(C-A-B)t-2(B-1)(1-t)]t(1-t)+\frac{1}{\mu}t(1-t)^2,$$

and $X_4(t)=t^2(1-t)^2$.

Since $\phi(1-t)>0$, $\phi'(1-t)>0$ and $\phi''(1-t)>0$ (see also [1, 3, 9]), proving $X_i(t)>0$ for $i=1,2,3$ gives $h_1(t)\geq 0$ and $h_2(t)\geq 0$ which will imply the required result for $0<t<1$.

Now $X_1(t)=\left[(C-A-B)t+\left(1+\frac{1}{\mu}-(B-1)\right)(1-t)\right]>0$, which clearly holds since $(C-A-B)>1$ and $B<2+\frac{1}{\mu}$ for all $t\in(0,1)$.

Similarly to prove $X_2(t)>0$, it is enough to prove

$$(B-1)\left(B-2-\frac{1}{\mu}\right)(1-t)+(C-A-B)\left(\frac{1}{\mu}-2(B-1)\right)t>0,$$

as the other term in $X_2(t)$ is positive on $0<t<1$. Since $B<1$ and $\frac{1}{\mu}>(B-2)$ from the hypothesis, the term involving $(1-t)$ is non-negative. Given that $\frac{1}{\mu}>(B-2)$, since $B<1$ we have $\frac{1}{\mu}>2(B-1)$ and $(C-A-B)>1$, which gives $X_2(t)>0$ for $0<t<1$.

Now proving $X_3(t)>0$ is equivalent to prove $2(C-A-B)t+\left[\frac{1}{\mu}-2(B-1)\right](1-t)>0$. Since $2(C-A-B)t+\left[\frac{1}{\mu}-2(B-1)\right](1-t)>0$, by hypothesis, and $\frac{1}{\mu}t(1-t)^2>0$ for $\mu>0$ and $0<t<1$, $X_3(t)>0$. ☐

Remark 3.4.

(1) *For the particular value of* $\xi=0$, *Theorem 3.4 yields a result with a smaller range for the parameters than the result given in* [1, Theorem.5.5].

(2) *For the case* $\xi=1$, *Theorem 3.4 results coincides with the result given in* [3, Theorem.5.8].

Theorem 3.5. *Let* $0\leq\xi\leq1$, $a>-1$, $b>-1$ *and* $\beta<1$ *satisfies* (1.13). If $f(z)\in W_\beta(\alpha,\gamma)$, then the function

$$V_\lambda(f)(z)=\int_0^1\lambda(t)\frac{f(tz)}{t}dt$$

where $\lambda(t)$ is given by

$$\lambda(t)=\begin{cases}(a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a}, & b\neq a,\\ (a+1)^2t^a\log(1/t), & b=a.\end{cases}$$

belongs to $M(\xi)$ if a, b and μ satisfies one of the following conditions:

(i) $b>a,\quad -1<a<0$ and $\quad b+a-1<\frac{1}{\mu}<b-1,$

(ii) $b<a,\quad -1<b<0$ and $\quad b+a-1<\frac{1}{\mu}<a-1,$

(iii) $b=a<0$ and $\frac{1}{\mu}>b-1$, which for $\mu\geq1$ (as in Theorem 2.2) gives $b<2\Rightarrow b<\min\{0,2\}=0$.

Proof. To prove the required result we need to show that (3.1) holds.
For the case $a\neq b$, $\lambda(1)=0$, $\Lambda_\nu(1)=0$ and $\lambda'(t)\leq0$. So it is enough to show the inequality (3.2). On substituting the values for $\lambda(t)$, $\lambda'(t)$ and $\lambda''(t)$, we need to show that $(1-\xi)A(t)+\xi B(t)\geq0$, where

$$A(t)=\frac{(a+1)(b+1)}{(b-a)}\left(\left[\left(1+\frac{1}{\mu}\right)-a\right]t^a-\left[\left(1+\frac{1}{\mu}\right)-b\right]t^b\right)$$

and

$$B(t)=\frac{(a+1)(b+1)}{(b-a)}\left(\left[a(a-1)-\frac{1}{\mu}a\right]t^a-\left[b(b-1)-\frac{1}{\mu}b\right]t^b\right).$$

So it is enough to prove $A(t)>0$ and $B(t)>0$ for $0<t<1$.

Case(i) $a<b$: Since $a>-1$ and $b>-1$, so $\frac{(a+1)(b+1)}{(b-a)}>0$. we need only to show that $\left[\left(1+\frac{1}{\mu}\right)-a\right]t^a-\left[\left(1+\frac{1}{\mu}\right)-b\right]t^b>0$, which clearly holds since $b>\left(1+\frac{1}{\mu}\right)$. Hence $A(t)>0$ for all $t\in(0,1)$. Now for $B(t)$ to be positive, it is enough to show that $\left(\left[a(a-1)-\frac{1}{\mu}a\right]t^a-\left[b(b-1)-\frac{1}{\mu}b\right]t^b\right)>0$ which is satisfied by the given condition on b.

Case(ii) $b < a$: Since $a > -1$ and $b > -1$, so $\dfrac{(a+1)(b+1)}{(b-a)} < 0$. We need only to show that $\left[\left(1+\dfrac{1}{\mu}\right) - a\right] t^a - \left[\left(1+\dfrac{1}{\mu}\right) - b\right] t^b < 0$ which is true since $a > \left(1+\dfrac{1}{\mu}\right)$. Now for $B(t)$ to be positive, it is enough to show that

$$\left(\left[a(a-1) - \frac{1}{\mu}a\right] t^a - \left[b(b-1) - \frac{1}{\mu}b\right] t^b\right) < 0$$

which is satisfied by the given condition on a.

Case(iii) $a = b \leq 0$: Changing inequality (3.2), which is true for $a = 0$. Hence we only consider the situation $a < 0$. Substituting the values of $\lambda(t)$, $\lambda'(t)$ and $\lambda''(t)$ in (3.2), an easy computation shows that for $0 \leq \xi \leq 1$, it suffices to show that the expressions

$$\log\left(\frac{1}{t}\right)\left(1+\frac{1}{\mu}-a\right)+1 \quad \text{and} \quad \left(a(a-1)-\frac{a}{\mu}\right)\log\left(\frac{1}{t}\right)+\left(\frac{1}{\mu}-2a+1\right)$$

are non-negative. Since $\log\left(\dfrac{1}{t}\right)$ is positive, the non-negativity of the first expression $\log\left(\dfrac{1}{t}\right)\left(1+\dfrac{1}{\mu}-a\right)+1$ follows from hypothesis (iii) of the theorem. Similar observation shows that the second expression reduces to $a(a-1)-\dfrac{a}{\mu} \geq 0$ and $\dfrac{a}{\mu} - 2a + 1 \geq 0$ using $\log\left(\dfrac{1}{t}\right)$ is positive. These two inequalities, for $a < 0$, gives $\dfrac{1}{\mu} \geq \max\{2a-1, a-1\} = a-1$, which is hypothesis (iii). The proof is complete. □

Theorem 3.6. *Let $c < 0$ and $\mu \geq 1$ with $0 \leq \xi \leq 1$. Further suppose that $p > 2$ and $\beta < 1$ be given by*

$$\frac{\beta - 1/2}{1-\beta} = -\frac{(1+c)^p}{\Gamma(p)} \int_0^1 t^c \left(\log\frac{1}{t}\right)^{p-1} q(t)dt,$$

where $q(t)$ satisfies (1.7). Then for the function $f(z) \in W_\beta(\alpha,\gamma)$,

$$V_\lambda(f) = \frac{(c+1)^p}{\Gamma(p)} \int_0^1 \left(\log\frac{1}{\gamma}\right)^{p-1} t^{c-1} f(tz)dt,$$

belongs to $M(\xi)$.

Proof. Choosing $\phi(1-t) = \left(\dfrac{\log(1/t)}{1-t}\right)^{p-1}$, we take $C - A - B = p-1$ and $B = c+1$ so that $\lambda(t)$ takes the form

$$\lambda(t) = Kt^c(1-t)^{p-1}\phi(1-t), \quad K = \frac{(1+c)^p}{\Gamma(p)}.$$

We complete the proof by applying Theorem (3.4) and using a simple computation to obtain $c < \min\{0, 1+\dfrac{1}{\mu}\} = 0$. □

4. A generalized integral operator

In this section for the functions $f \in W_{\beta}(\alpha, \gamma)$, we consider another integral operator introduced in [4] and find the admissibility conditions to be in the class $M(\xi)$.

Theorem 4.1. *Let $\mu > 0$, $\nu > 0$, satisfies* (1.2)*, then for $\rho < 1$ and $\beta < 1$ satisfying*

$$\frac{1}{2(1-\beta)(1-\rho)} = \int_0^1 \lambda(t)\left[(1-\xi)\left(\frac{1-g(t)}{2}\right) + \xi(1-q(t))\right] dt, \tag{4.1}$$

where $g(t)$ and $q(t)$ are defined by (1.6) *and* (1.8) *respectively. Assume that for $f \in \mathcal{A}$,*

$$\mathcal{V}_{\lambda}(f)(z) = z\int_0^1 \lambda(t)\left(\frac{1-\rho t z}{1-tz}\right) dt * f(z),$$

then $F = \mathcal{V}_{\lambda}(W_{\beta}(\alpha, \gamma)) \subset M(\xi) \Longleftrightarrow N_{\Pi_{\mu,\nu}}(h) \geq 0$.

Proof. The proof follows similar lines of proof of Theorem 2.1. Hence, we omit details. □

Note 2. *$\mathcal{V}_{\lambda}(f)(z) = \rho z + (1-\rho)V_{\lambda}(f)(z)$ and hence this operator generalize the operator given in* (1.1)*.*

For finding applications of the operator $\mathcal{V}_{\lambda}(f)(z)$, by virtue of (2.2), Theorem 3.2 is ssufficient. This means we use the conditions given in Section 3. Hence we state the following results without giving their proof as they can be obtained in a similar fashion as in the results of Section 3.

Corollary 4.1. *Let $0 \leq \xi \leq 1$, $\nu > 1$, $\mu > 1$, satisfies* (2.1)*, Then for $\rho < 1$ and $\beta < 1$ satisfying*

$$\frac{1}{2(1-\beta)(1-\rho)} = (c+1)\int_0^1 t^c\left[(1-\xi)\left(\frac{1-g(t)}{2}\right) + \xi(1-q(t))\right] dt$$

where $g(t)$ and $q(t)$ are defined by (1.6) *and* (1.8) *respectively. Assume that for $f \in W_{\beta}(\alpha, \gamma)$, the function $\mathcal{V}_{\lambda}(f)(z)$ belongs to $M(\xi)$ provided*

$$-1 < c \leq min\left[\left(1+\frac{1}{\mu}-\frac{1}{\nu}\right), \left(\frac{1+\dfrac{1}{\mu}-\xi}{1+2\xi}\right)\right].$$

Corollary 4.2. *Let $0 \leq \xi \leq 1$, $\nu > 1$, $\mu > 1$, satisfies* (2.1)*, Then for $\rho < 1$ and $\beta < 1$ satisfying*

$$\frac{1}{2(1-\beta)(1-\rho)} = k\int_0^1 t^{B-1}(1-t)^{C-A-B}\phi(1-t)\left[(1-\xi)\left(\frac{1-g(t)}{2}\right) + \xi(1-q(t))\right] dt$$

where $g(t)$ and $q(t)$ are defined by (1.6) *and* (1.8) *respectively. Assume that for $f \in W_{\beta}(\alpha, \gamma)$, the function $\mathcal{V}_{\lambda}(f)(z)$ belongs to $M(\xi)$ provided*

$$B < min\left(\left[2+\frac{1}{\mu}\right], (C-A-1)\right).$$

Corollary 4.3. *Let $0 \leq \xi \leq 1$, $a > -1$, and $b > -1$. Then for $\rho < 1$ and $\beta < 1$ satisfying*

$$\frac{1}{2(1-\beta)(1-\rho)} = \int_0^1 \lambda(t)\left[(1-\xi)\left(\frac{1-g(t)}{2}\right) + \xi(1-q(t))\right] dt,$$

where $g(t)$ and $q(t)$ are defined by (1.6) *and* (1.8) *respectively and $\lambda(t)$ is given by*

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a}, & b \neq a, \\ (a+1)^2 t^a \log(1/t), & b = a. \end{cases}$$

Assume that for $f \in W_\beta(\alpha, \gamma)$, the function $\mathcal{V}_\lambda(f)(z)$ belongs to $M(\xi)$ provided

(i) $b > a, \quad -1 < a < 0$ *and* $\quad b + a - 1 < \dfrac{1}{\mu} < b - 1,$

(ii) $b < a, \quad -1 < b < 0$ *and* $\quad b + a - 1 < \dfrac{1}{\mu} < a - 1,$

(iii) $b = a < 0$ *and* $\dfrac{1}{\mu} > b - 1$, *which for* $\mu \geq 1$ (as in Theorem 2.2) *gives* $b < 2 \Longrightarrow b < \min\{0, 2\} = 0.$

Corollary 4.4. *Let $c < 0$, $\mu \geq 1$ satisfies* (2.1), *with $0 \leq \xi \leq 1$ Then for $p > 2$, $\rho < 1$ and $\beta < 1$ satisfying*

$$\frac{1}{2(1-\beta)(1-\rho)} = \frac{(1+c)^p}{\Gamma(p)} \int_0^1 t^c \left(\log \frac{1}{t}\right)^{p-1} \left[(1-\xi)\left(\frac{1-g(t)}{2}\right) + \xi(1-q(t))\right] dt$$

where $g(t)$ and $q(t)$ are defined by (1.6) *and* (1.8) *respectively. Then for $f \in W_\beta(\alpha, \gamma)$, the function $\mathcal{V}_\lambda(f)(z)$ belongs to $M(\xi)$.*

Remark 4.1. *All the above applications at $\rho = 0$ reduces to the results obtained in Section* 3. *Further Corollary* 4.1 *at $\xi = 1$ and $\xi = 0$ reduces respectively to corollaries* 6.4 *and* 6.5 *given in* [3]. *All the other corollaries in this section for $\rho \neq 0$ and $0 < \xi < 1$ are not discussed in the literature elsewhere.*

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